FIRST YEAR UNDERGRADUATE STUDENTS' ERRORS WHEN SOLVING INEQUALITIES

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The purpose of this study was to investigate errors of first year undergraduate students when solving inequalities. Data were generated through a test and follow-up interviews. The inequalities include linear, quadratic, rational and absolute value inequalities. Findings reveal errors in expressing compound inequalities, errors in representing solutions as intervals, generalising equality to inequality, including a zero of denominator in the solution set of rational inequalities, generalising the definition of absolute value and incomplete solution processes. The results also show that there was one comprehension error and that the other errors occurred because of problems in transformation, process skills and encoding solutions. The findings suggest that the errors in expressing compound inequalities arose because the students did not understand underlying concepts. For instance, most students ignored the idea of numerical order implicit in inequalities. The results also show that the errors in expressing compound inequalities occurred because of the nature of the inequalities, the lack of a formal definition of absolute value and the lack of proofs for properties of absolute inequalities. The results suggest need for teacher educators to focus on these errors and the underlying concepts of inequalities in order to enhance students' conceptual understanding of inequalities. The results also offer a valuable opportunity for deliberations on how to prevent, limit and deal with such students' errors.

INTRODUCTION

There is wide agreement that it is important for educators to be familiar with their students' ways of thinking about mathematical concepts, both correct and incorrect, and to be aware of possible causes of their students' errors and misconceptions (Ball, Thames & Phelps, 2008). Such knowledge significantly contributes to the instruction process (Even & Tirosh, 2002; Ball et al., 2008). The creation of a body of knowledge about ways of thinking, errors and misconceptions, and previous conceptions that students have about the subjects they learn at school can contribute to educators' pedagogical content knowledge (Shulman, 1986). The educator who takes into account his/her students' knowledge and ways of thinking can develop appropriate activities for them (Even & Tirosh, 1995) and can implement a teaching model which emphasises conceptual understanding.

Understanding the principles and usage of inequalities is also of fundamental importance in Mathematics. An inequality is a mathematical sentence built from expressions using one or more of the symbols ($<$, $>$, $\leq$ or $\geq$) to compare two quantities. Solving inequalities means finding the value(s) of unknowns that make the relationship true. Inequalities occupy an important place in the basic mathematics concepts, and are an important entry point for a lot of Mathematical topics such as equations and different kinds of functions (Anggoro & Prabawanto, 2019). Inequalities are important in studying properties and applications of functions which require students to understand methods of finding the solution set of different types of inequalities (linear, quadratic rational and absolute value inequalities). Inequalities also provide a complementary perspective to equations. At university level, the concept of inequalities forms part of the foundation for Mathematics and Engineering Science.

In light of the above, the educator's knowledge and understanding of students' errors when solving inequalities will help to develop strategies in teaching that address students' errors and misconceptions. Boero and Bazzini (2004) agree that the concept of inequalities is "an important subject from the mathematical point of view but a difficult subject for students and a subject scarcely considered till now by researchers in mathematics education". An awareness of the issue that inequalities were not under the microscope of the mathematics educators' community emerged five years prior to Boero and Bazzini's (2004) statement, when the Project Group at the 23rd Psychology of Mathematics Education Conference called for research on inequalities (Halmaghi, 2011). A lot of previous studies conducted so far reveal that students often have difficulties and misconceptions when dealing with mathematics inequality problems (Anggoro & Prabawanto, 2019). Malawi National Examinations Board (MANEB) chief examiners' report for Mathematics (2017) agrees that students have difficulties with solving inequalities. However, no study has been conducted in Malawi to support the previous research studies.

Thus this study was conducted to put to light beginning university students' errors when solving inequalities and the possible sources of the errors with the aim of addressing the gap in global literature and the teaching of inequalities in Malawi. The study answered the following research question: What errors do undergraduate Algebra students commit when solving inequalities? (2) At what stage of inequality solving process do the students commit such errors? Results from the study could suggest the need for educators to focus on the errors and the underlying concepts of inequalities in order to enhance students' conceptual understanding of inequalities.

CONCEPTUAL FRAMEWORK

The conceptual framework used in this study is based on Newman Error Analysis Model proposed by Newman (1977). This model has proved to be a reliable model for Mathematics educators. It has six types of errors: reading, comprehension, transformation, process skill, encoding and carelessness which form a hierarchy that classifies types of errors based on the problem solving level of students. Prakitipong and Nakamura (2006) point out that in the process of problem solving there are two kinds of obstacles that hinder students from arriving at correct solutions: (a) Problems in linguistic fluency and conceptual understanding that correspond with level of simple reading and understanding meaning of problems, and (b) Problems in mathematical processing that consists of transformation, process skills, and encoding. This classification implies that the students have to interpret the meaning of the questions before they proceed to mathematical processing to obtain appropriate solutions. The conceptual framework is shown in Figure 1.

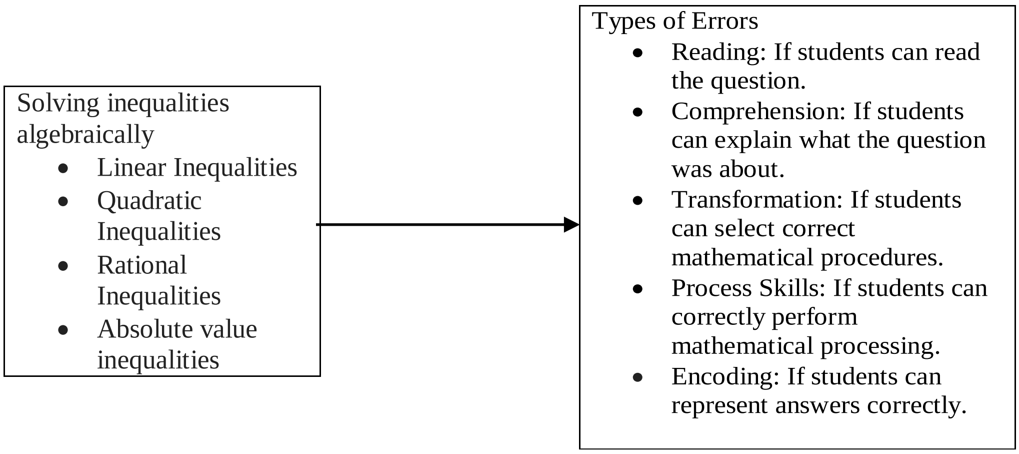


Figure 1: Conceptual framework for the study developed by researcher

Newman's Error Analysis (NEA) provided a framework for considering the reasons that underlay the difficulties students experienced with mathematical problems and a process that assisted to determine where errors occurred. NEA also provided directions for where educators could target effective teaching strategies to overcome students' errors.

Researchers who made use of this framework include Prakitipong and Nakamura (2006), Zakaria (2010), Chinamasa, Nhamburo and Sithole (2014) and Ragma (2014). All of their studies were able to identify the specific error categories and reasons for the errors of their respondents. What is not yet clear is the type and level of errors undergraduate students make when solving inequalities. This study extends the application of Newman's (1977) error analysis to undergraduate students' learning of inequalities.

Students' errors and misconceptions when solving inequalities

Various studies reveal a number of difficulties students display when solving inequalities within the various stages of education. First, students regard inequalities as equations (Bicer, Capraro & Capraro, 2004; Tsamir & Bazzini, 2004). Second, El-Khateeb (2016) also found that students made errors in basic algebraic operations and deletion, when solving fractional inequalities and some errors involved absolute value inequalities. Third, the most common difficulty students possess is the interpretation of solution sets (Blanco & Garrote, 2007; Çiltaş & Tatar, 2011; Halmaghi, 2011). Çiltaş and Tatar (2011) also observed that students experienced difficulties in solving absolute value inequalities by acting as if there were no absolute values. Results from the study also showed that students experienced difficulties in interpreting intervals that were found correctly in their solution processes. At university level, previous studies also show students errors in representing the solution set as an interval or on the number line, multiplying/dividing by a negative number and arithmetic errors (Üreyen, Mahir & Çetin, 2006). Anggoro and Prabawanto (2019) also investigated undergraduate students' conceptual understanding on rational inequalities. Results revealed a lot of errors and misconceptions especially on translating from graphs to algebraic symbolism and vice versa.

Various researchers have attributed students' errors to several sources (Luneta & Makonye, 2010; Muzangwa & Chifamba, 2012; Chinamasa et al., 2014). What all these writers agree on is that students commit errors which are a result of existing conceptual gaps or misconceptions which are entrenched in the students' conceptual schemes (Godden, Mbekwa & Julie, 2013). For instance, students often passively accept the rules and techniques of formal calculation without having any control of the meaning conveyed by these rules or of the properties on which these techniques are based. In addition, El-Shara' and Al-Abed (2010) found that students' difficulties with inequalities can be attributed to three sources such as the nature of the subject, the students themselves and the teacher whose responsibility is to reduce the effect of each of the sources. Tsamir and Reshef (2006) emphasise that instructional approaches used by teachers may have an effect on the number of students' mistakes and their nature.

All in all, the review of previous literature indicates that globally, research reports on students' errors when solving inequalities are scanty and that nothing is known about the nature of errors Malawian undergraduate students make when solving inequalities. In addition, studies on undergraduate students' error analysis using NEA model are less available. Furthermore, none of the literature sources reviewed indicate the nature and the stage at which the errors are made during mathematical problem solving. As such, this study was intended to understand the nature of undergraduate students' errors when solving inequalities and the problem solving stage at which the errors are made.

RESEARCH METHOD

This study is located within the interpretative qualitative research paradigm. Qualitative research is an exploratory approach, which emphasises the use of open-ended questions and probes, giving participants an opportunity to respond in their own words (Merriam, 2009). The study investigated first year undergraduate students' errors when solving inequalities. The sample comprised 48 purposefully selected Bachelor of Technical Education students at the Malawi Polytechnic because they had just learnt the topic three weeks prior to the study. Five test items were prepared and the questions were adapted from textbooks. The following are the inequalities which the students solved during the test: $4x - 3 < 2x + 5$, $|2x - 3| > 9$, $\frac{1}{|2x-1|} < \frac{1}{|x-2|}$ and $\frac{x^2+4x+3}{x-1} > 0$. By the time the students were writing the test, they had practiced solving these inequalities during class exercises, homework exercises and assignments as well as tutorials. A follow up interview was administered a week after the students had written the test and it was conducted if the respondent failed to answer correctly in their written test.

Data were analysed using thematic analysis (Ritchie, Spencer, & O'Connor, 2003) and basic frequency counts. Themes were developed *a-priori* from the conceptual framework and *a-posteriori* from the data. The themes obtained inductively include generalisation of equality to inequality, absolute value inequality errors, misrepresentation of inequalities into compound inequalities, inclusion of zeros of denominator in the solution set when solving rational inequalities and inability to represent solutions of quadratic inequalities. After the analysis of the test, interviews were conducted with six students to find out the reasons behind the errors they made. The interviews were transcribed in whole and the transcripts were analysed for reasons behind the students' errors. Thereafter, the identified errors were also analysed using the NEA model to find out the level at which the students made the errors. The errors were classified into comprehension errors, transformation errors, process skill errors and encoding errors. This was done because inductive thematic analysis helped me to understand the nature of the errors students displayed and the NEA model helped to understand the problem solving stage at which the errors were made. To achieve credibility of the results, 10% of the test papers were analysed by another independent researcher. In all cases, we achieved 90% agreement without discussion between the two of us.

RESULTS AND DISCUSSION

The results show that the students made seven errors and the errors relate to problems in comprehension and mathematical processing (Newman, 1977; Prakitipong & Nakamura, 2006). The errors occurred because the students had problems with transformation, process skills, and encoding solutions. These errors are presented in Table 1 together with the frequencies of occurrence and are then discussed.

Table 1: Errors identified during the test.

Error	Type	Frequency
Errors in expressing compound inequalities	Transformation, process skills error	31
Errors in representing the solution as interval	Encoding error	29
Generalisation of equality to inequality	Process skills, encoding error	23
Inclusion of zeros of denominator in the solution set when solving rational inequalities	Encoding error	33
Incomplete solution processes	Transformation, encoding error	11
Overgeneralisation of $\lvert x\rvert = x$	Comprehension, Transformation	4

Table 1 shows that over half of the students made the absolute value inequality errors and inclusion of zeros on the denominator from the solution set when solving rational inequalities. The results show that 31 students made the absolute value inequality error and 14 made errors in solutions of quadratic inequalities while 29 had difficulties with representing solutions as intervals. Generalisation of equality to inequality appeared twenty three times and 11 had incomplete solution processes. Analysis showed that a student could commit more than one error. The results in Table 1 also show that none of the students made errors related to multiplying/dividing by a negative number, particularly when solving the inequality $\frac{1}{|2x-1|} < \frac{1}{|x-2|}$ where students had to transform $-3x^2 + 3 < 0$ into $3x^2 - 3 > 0$ and also when solving $-3x^2 < -21x + 30$. This is contrary to the results of studies conducted by Bicer et al. (2014).

Errors in expressing compound inequalities

An absolute value inequality is a problem with absolute values as well as inequality signs. For the inequality $|2x - 3| > 9$, three errors were identified. First, some students broke the absolute inequality into $2x + 3 > 9$ or $2x + 3 > -9$ instead of $2x + 3 < -9$ (Figure 2).

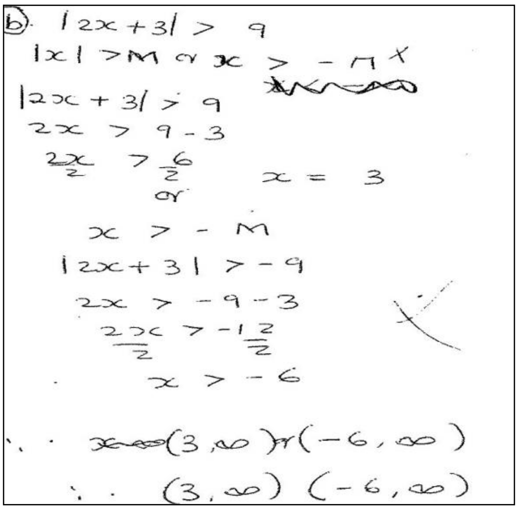


Figure 2: Tapempha's error in expressing a compound inequality

Results in Figure 2 reveal that the students who made this error think that since $|2x + 3|$ is greater than 9, then the two inequalities should have a > (greater than) symbol. In fact, they were unable to make sense of the fact that having the inequalities $2x + 3 > 9$ and $2x + 3 > -9$ which led to $x > 3$ and $x > -6$ include the values between -6 and 3 which are not elements of the solution set. This is a processing skill error and might have stemmed from the fact that any students struggle with solving absolute value inequalities because they do not have an understanding of the underlying concepts.

Second, the students misrepresented inequalities into compound inequalities. They joined $-6 > x$ or $x > 3$ into $-6 > x > 3$. After getting the solutions $-6 > x$ and $x > 3$, the students joined the two inequalities into a compound inequality because they failed to make sense of the impossible case $-6 > x > 3$ (Figure 3). This indicates problems encoding answers.

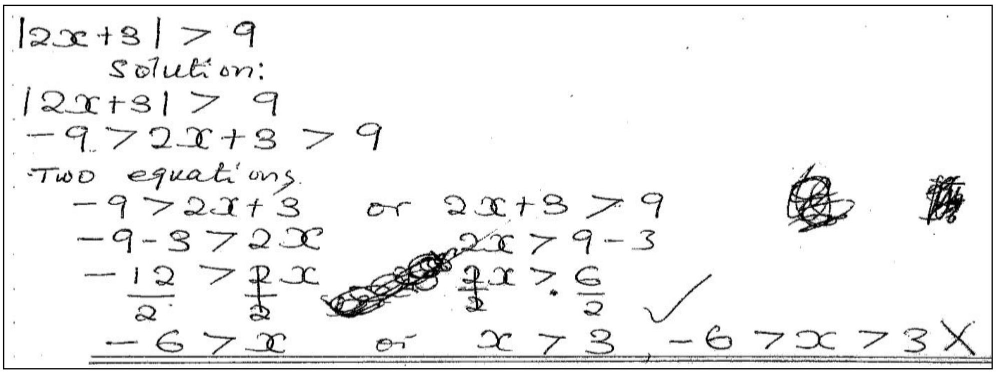


Figure 2: Pemphero's error in expressing a compound inequality

The third error that the students made was when transforming $|2x - 3| > 9$ into $2x - 3 < -9$ or $2x - 3 > 9$. Some of them came up with an incorrect compound inequality

$-9 < 2x - 3 < 9$ (see Figure 4). In this case, the students incorrectly used $|x| < M \Leftrightarrow -M < x < M$ (property 1) instead of $|x| > M \Leftrightarrow x < -M$ or $x > M$ (property 2). Thus the students solved the problem as one inequality probably because they did not understand that the case $|x| > M$ gives rise to two inequalities $-M < x$ and $x > M$ during the inequality transformation process. In this case M is any positive real number.

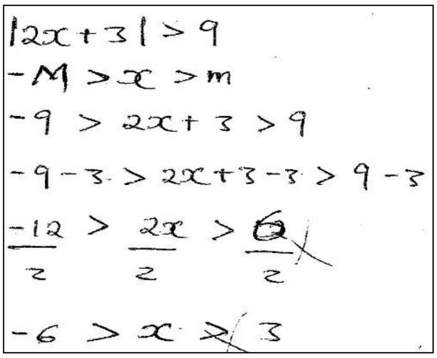


Figure 3: Chisomo's error in expressing a compound inequality

Figure 4 shows that the students had difficulties conceptualising the property that $|x| > M$ has two distinct solutions. They focused on what they had previously learnt in secondary schools that the solution of the inequality has a most expected form of an inequality $-M < x < M$ (Tsamir & Reshef, 2006).

The results might also have come out like this because first, the concept of absolute value is introduced informally using the metaphor of distance and supplemented with the "property", if M is a positive number, then $|x| = M$ is equivalent to $x = M$ or $x = -M$ (property 0). There is no formal definition. This definition often initially confuses students because they think the expression $-M$ is negative which goes against their understanding that the absolute value of a quantity is positive. Students with this misconception do not understand that $x = -M$ is a critical part to this definition (Ponce, 2008). It is a critical part because if x is less than 0, then x is to the left of 0 on the number line. Second, properties 1 and 2 used to solve absolute inequalities are not proved because there is no definition to found the proof on (Sierpinska, Bobos, & Pruncut, 2011). The solution process techniques for $|x| < M$ and $|x| > M$ are demonstrated on 'worked out examples' and this makes the concept hard to understand.

> The technique conceals the logical symmetry of the two types of inequalities: there is conjunction of conditions in one inequality and disjunction in the other. If the properties on which the techniques are based were proved, or if the inequalities were solved using a definition of absolute value and Reasoning by Cases, a common pattern of reasoning would be revealed, using disjunction in both inequalities and this could open a way to generalisation (Sierpinska et al., 2011, p. 283).

For instance, using the mentioned definition (property 0), the proofs of Properties 1 and 2 would exhibit reasoning by cases, which always involves disjunction of the conditions describing the cases:

$$|x| < a \Leftrightarrow (x \geq 0 \text{ and } x < a) \text{ or } (x < 0 \text{ and } -x < a) \Leftrightarrow x \in [0, a) \cup (-a, 0) \Leftrightarrow (-a, a)$$

$$|x| > a \Leftrightarrow (x \geq 0 \text{ and } x > a) \text{ or } (x < 0 \text{ and } -x < a) \Leftrightarrow x \in [a, \infty) \cup (-a, 0)$$
$$\Leftrightarrow (-\infty, -a)$$

(Sierpinska, et al., 2011).

These results concur with the findings of study conducted by Zakaria et al. (2010) which revealed that most errors made were transformation and process skills errors.

Generalisation of equality to inequality

This is an encoding error in which students showed that they did not understand the difference between equation and inequality and thus failed to interpret the resulting solution. Figures 2, 3 and 5 illustrate the students' use of the equality symbol in place of an inequality symbol. It is worth noting that generalising equality to inequality occurs in all forms of inequalities. In Figure 2, Tapempha used the relation $x = 3$ when simplifying $2x > 6$. In Figure 3, Pempho wrote 'two equations' to refer to 'two inequalities'. This error signifies that the students were unable to represent the result from mathematical processing because they regarded solving inequalities as a process of producing an answer so that $x = -6$ or $x = 3$. In addition, students tend to solve inequalities using algorithmically algebraic transformations that are suitable when solving equations but they ignore the special properties of inequalities (Bazzini & Tsamir, 2001). Furthermore, this error implies that students are taught to solve inequalities in the same way equations are solved but with certain exceptions. This approach is mathematically problematic because it ignores the idea of equivalence implicit in equations and the idea of numerical order implicit in inequalities. These results were also found by Bicer et al. (2014).

Another form of students' generalisation of equality to inequality was identified in solution processes for the absolute value inequality $\frac{1}{|2x-1|} < \frac{1}{|x-2|}$, of which the solution process was quadratic in nature. After cross-multiplying and squaring on both sides, the students came up with a quadratic inequality which they simplified and arrived at the inequality $x^2 - 1 > 0$ which they transformed into $(x+1)(x-1) > 0$. Then they came up with $(x+1) > 0$ or $(x-1) > 0$ and finally arrived at $x > 1$ or $x > -1$ (Figure 5) as if they were solving quadratic equations.

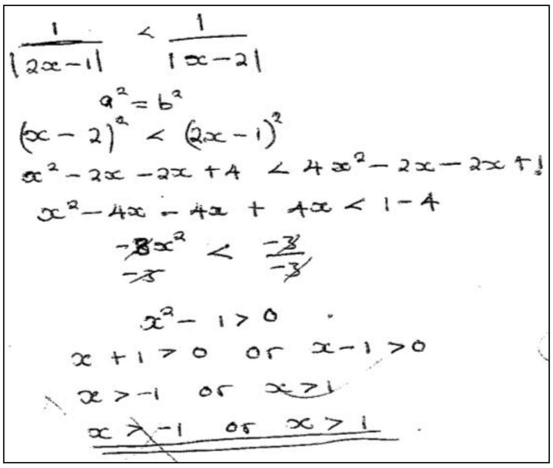


Figure 5: Promise's generalisation of equality to inequalities

Promise's error in Figure 5, might have originated from incorrect application of quadratic equation structure to inequalities. After obtaining $x^2 - 1 > 0$, the students worked out the inequality the way they would solve a quadratic equation. This is an incorrect generalisation of the technique they use in solving quadratic equations. Students who are unaware of the why behind the strategy used in solving quadratic equations are likely to commit this error (Naseer, 2015). It is also evident that Promise and other students who made this error did not understand the circumstances under which the inequality sign changes to opposite. In particular, they did not understand that to solve $(x+1)(x-1) > 0$, students must consider two cases. Case 1: $x+1 > 0$ and $x-1 > 0 \Rightarrow x > -1$ and $x > 1$. In this case, the solution is $x > 1$. Case 2: $x+1 < 0$ and $x-1 < 0 \Rightarrow x < -1$ and $x < 1$. In this case, the solution is $x < -1$. So the solution set is $(-\infty, 1) \cup (1, \infty)$. To obtain these ranges of values, students encounter two difficulties; one of the cases may be forgotten, or it may be difficult to select the correct solution from $x > -1$ and $x > 1$ and from $x < -1$ and $x < 1$ (MacLaurin, 1985). Thus the gap in knowledge caused the students who made this error to fail to represent the solution for $(x+1)(x-1) > 0$. Tsamir and Bazzini (2001) also found that many high school students' responses to $(y-2)(y+9) > 0$ were $(y-2) > 0$ and $(y+9) > 0$.

Inclusion of zeros of the denominator in the solution set when solving rational inequalities

The results in Table 1 show that this error was made by 33 students in the solution processes of $\frac{x+1}{x+3} \leq -2$. It is an encoding error because the students failed to understand and put in order the range of values which make the inequality true (Newman, 1977; Zakaria, 2010). Figure 6 illustrates the error made by one of the students, Ella. Ella and other students who made this error solved the inequalities correctly but they incorrectly represented the range of values that make each rational inequality true, particularly on the denominator.

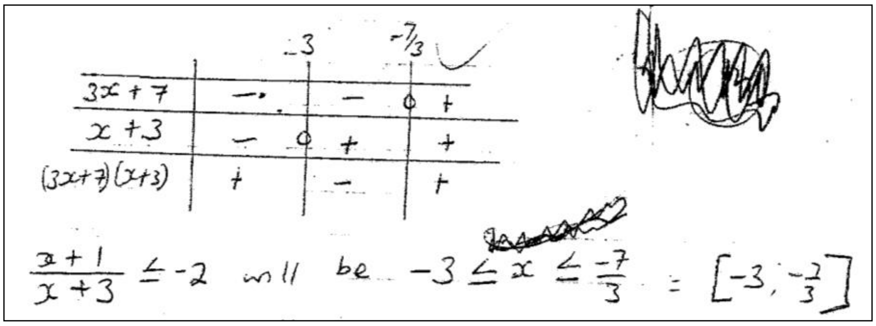


Figure 6: Ella's inclusion of a zero of denominator

In Figure 6, Ella wrote $-3 \leq x \leq -\frac{7}{3}$ and $[-3; -\frac{7}{3}]$ while others wrote $x \geq -3$ and $x \leq -\frac{7}{3}$ as solutions of the inequality. In these cases, Ella made $x = -3$ which makes the denominator equal to zero and therefore the whole inequality undefined. In fact, the solution of the inequality is in the interval $-3 \leq x \leq -\frac{7}{3}$ or it could be presented as $x > -3$ and $\leq -\frac{7}{3}$.

In their solution processes of the rational inequality $\frac{x^2+4x+3}{x-1} > 0$, the students who made this error included $x = 1$ in the solution set Figure 7 illustrates the error made by Davie.

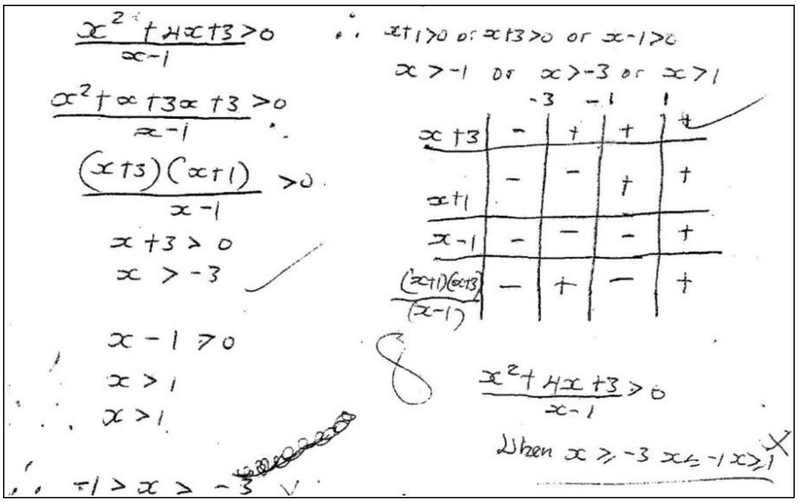


Figure 7: Davie's inclusion of a zero of the denominator

Ella and Davie's errors in Figures 6 and 7 error might imply that since the students had just started their Algebra course, it might be that they had not yet conceptualised and internalised the rule that when the denominator is zero, the inequality is undefined. This error was also identified by El-Khateeb (2016).

Interchanging inequality and equality signs

Figure 8 shows that other than generalising equality to inequality, some students used $=$ and $\leq$ or $\geq$ interchangeably probably because the process of solving equations is the same as that for solving inequalities. Dalitso wrote $\frac{3x+7}{x+3} = 0$ instead of $\frac{3x+7}{x+3} \leq 0$.

Errors in the use of logical connectives

This involves inappropriate use or "and" or "or" when giving solutions of inequalities. The results from the study also show that the undergraduate students in this study made errors in their choices of the logical connectives. Figure 8 illustrates that Dalitso was unable to choose elements of the solution set from the sign chart.

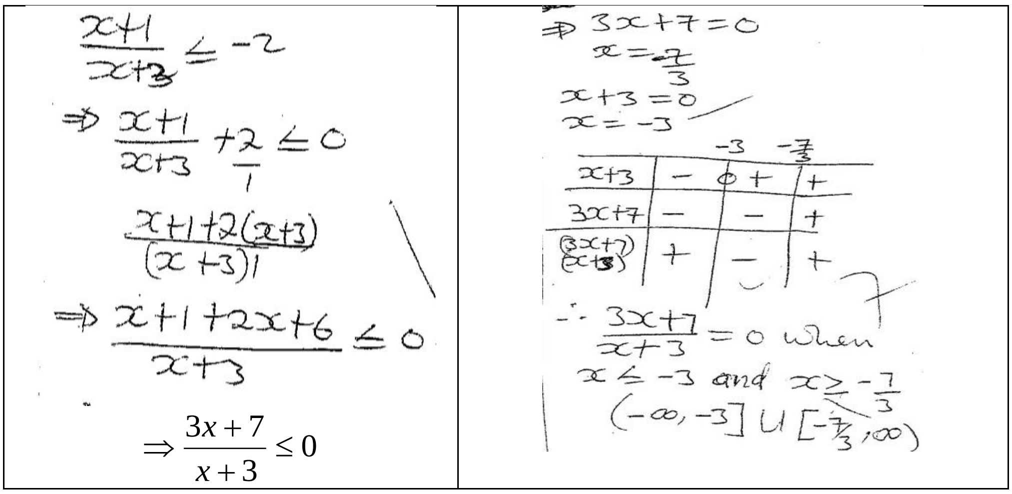


Figure 8: Dalitso's incorrect choices of logical connectives

This error resulted from the students' inability to comprehend and encode the use of logical connectives. Dalitso wrote $x \leq -3$ and $x \geq -\frac{7}{3}$. In this case, the challenge for Dalitso was to understand and encode that when the solution lies between two values, "and" is used, such that the range of values that satisfy the inequality is $-3 < x \leq -\frac{7}{3}$. Although the symbol $\cup$ was correctly used in this case, there is an inconsistency between Dalitso's $x \leq -3$ and $x \geq -\frac{7}{3}$ and $(-\infty, -3) \cup [-\frac{7}{3})$ because he used "and" incorrectly and the ranges of values do not represent the solution set. These findings reflect the findings of studies conducted by Tsamir and Almog (2001; Tsamir et al., 2004).

Errors in representing the solution set as an interval

Results in Table 1 also show that when solving rational inequalities, 10 students were not able to write their solutions in interval notation. They wrote closed intervals instead of open intervals and vice versa. They also reversed the numbers inside the intervals such as $(-3, -\infty), (-1, -\infty)$ or $(-6, -\infty)$. This error resulted from the students' inability to comprehend and encode the meaning of solutions of inequalities in interval notation. Figure 9 illustrates this error.

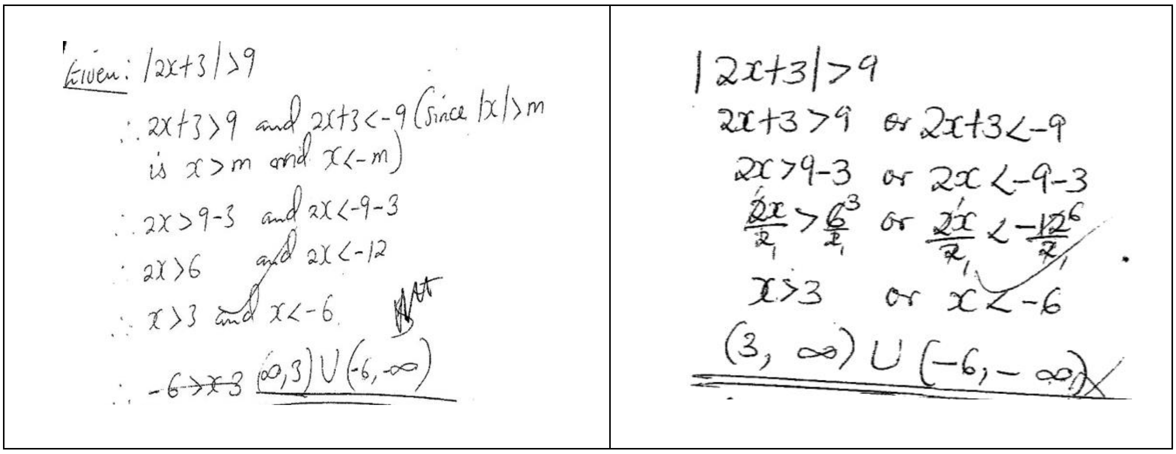
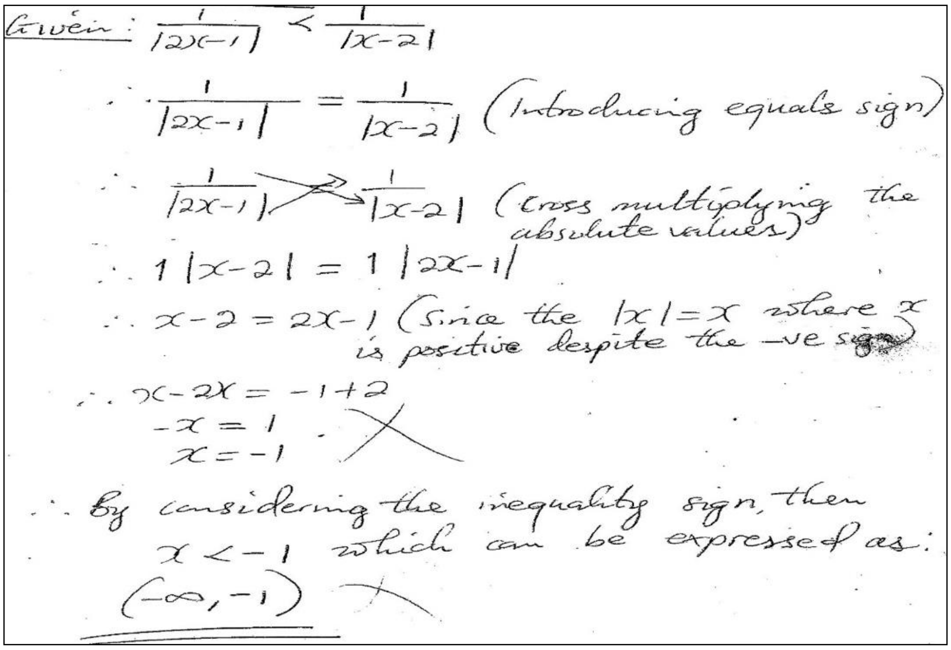


Figure 9: Students' errors in representing solutions as intervals

In Figure 9, in the solution process to $|2x + 3| > 9$, the student wrote the intervals $(\infty, 3) \cup (-6, -\infty)$ and in the second solution process, the student wrote $(3, \infty) \cup (-6, -\infty)$. This signifies the student's difficulties to relate inequalities and intervals. These results also show that some students had difficulties with the use of finite and infinite intervals especially when solving rational inequalities. In such cases, they included the zeros of the denominators as elements of closed intervals. This error might also have resulted from students' inability to understand the idea of numerical order implicit in inequalities.

Overgeneralisation of $|x| = x$

The results show that some students overgeneralised the definition of absolute value. Figure 10 illustrates that some students think that we obtain x from $|x|$ by just ignoring the absolute value notation such that $|x| = x \Longrightarrow |2x - 1| = 2x - 1$ but this is not true because $|2x - 1|$ is always positive while $2x - 1$ is not always positive. The results show that it is a comprehension error because the students did not understand what the question required them to do.


Given: $\frac{1}{|2x-1|} < \frac{1}{|x-2|}$

$\therefore \frac{1}{|2x-1|} = \frac{1}{|x-2|}$ (Introducing equals sign)

$\therefore \frac{1}{|2x-1|} \quad \frac{1}{|x-2|}$ (Cross multiplying the absolute values)

$\therefore 1\,|x-2| = 1\,|2x-1|$

$\therefore x-2 = 2x-1$ (Since the $|x|=x$ where x is positive despite the $-$ve sign)

$\therefore x-2x = -1+2$

$-x = 1$

$x = -1$

$\therefore$ By considering the inequality sign, then $x < -1$ which can be expressed as:

$(-\infty, -1)$

Figure 10: Joy’s generalisation of $|x| = x$

As Joy explains in his solution process in Figure 10, he introduced an equality symbol, cross multiplied to obtain $x - 2 = 2x - 1$. At this stage, the student shows misunderstanding of the theorem that if $x \in \mathbb{R}$, then $|x| = \sqrt{x^2}$ and that $|x|^2 = x^2$ which implies that after cross-multiplying, $|x - 2| < |2x - 1| \Leftrightarrow (x - 2)^2 < (2x - 1)^2$ produces a quadratic inequality $3x^2 - 3 < 0$ of which the solutions are $x < -1$ or $x > 1$. In this extract, Joy reveals the misconception that since $|x| = x$, then $|x - 2| = x - 2$ and similarly, $|2x - 1| = 2x - 1$ for all values of $x \in \mathbb{R}$ because he substituted the inequality symbol with equality. However, this inequality is not only true for only positive values of x. Also his equation $x - 2 = 2x - 1$ is not true for all positive values of x because if $x = 1,\ x - 2 = -1$ and $2x - 1 = 1$ which makes $x - 2 = 2x - 1$ false. The absence of meaning of absolute inequalities and solutions of inequalities is one of the principal problems that might have caused the error. Figure 10 also shows interference of previous knowledge to current knowledge (Radatz, 1980). The results show that Joy’s knowledge of solving linear equations interfered with his knowledge of solving inequalities and this made him ignore the absolute value notation. This illustrates the fact that errors and misconceptions influence each other.

Incomplete solution processes

Figure 11 is an example of a student’s incomplete solution process. After transforming the inequality $\frac{x+1}{x+3} \leq -2$ to $\frac{3x+7}{x+3} < 0$, the student did not find the range of values for which $3x + 7, x + 3$, and $\frac{3x+7}{x+3}$ are positive or negative. Other students found the range of values for which the expressions are positive or negative but were unable to find the solution set for the inequality from their sign charts.

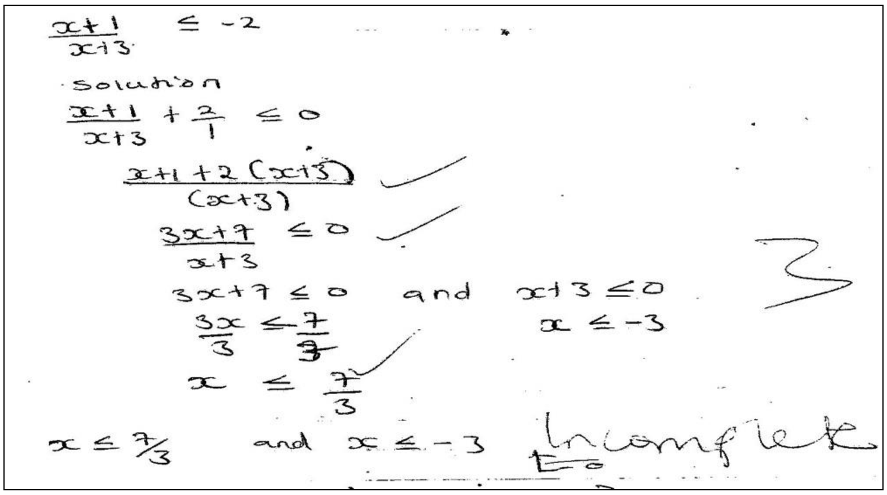


Figure 41: Maya's incomplete solution process

Results from the follow up interview show that although Maya expressed the inequality as a single quotient and got the zeros of the function as well as where the function is undefined, she did not understand that the zeros of $\frac{3x+7}{x+3}$ and the point where it is undefined are used to separate the number line into two intervals so as to determine the range of values where $\frac{3x+7}{x+3}$ is positive or negative.

Interviewer:	Why did you not finish solving the inequality?
Maya:	I forgot how to get the plus or minus from the table.

Clearly, Maya's interview transcript indicates that her error was a result of comprehension problems about how to come up with the sign chart and inability to internalise it. This indicates that Maya made a transformation error. Apart from the error illustrated in Figure 11, other students found it hard to determine the sign of the quotient and they had incomplete solution processes as well. When solving the rational inequality, $\frac{x^2+4x+3}{x-1} > 0$, after factoring $x^2 + 4x + 3$, some students developed and used the sign chart correctly to determine the regions where $x + 3$ and $x + 1$ are positive or negative and also $(x + 3)(x + 1)$ but forgot to find the regions where $\frac{(x+3)(x+1)}{x-1}$ is positive or negative. Other students managed to determine the range of values for which $\frac{(x+3)(x+1)}{x-1}$ is positive or negative but they were unable to pick the ranges of values from the sign chart. Figure 12 is Yamikani's incomplete solution process.

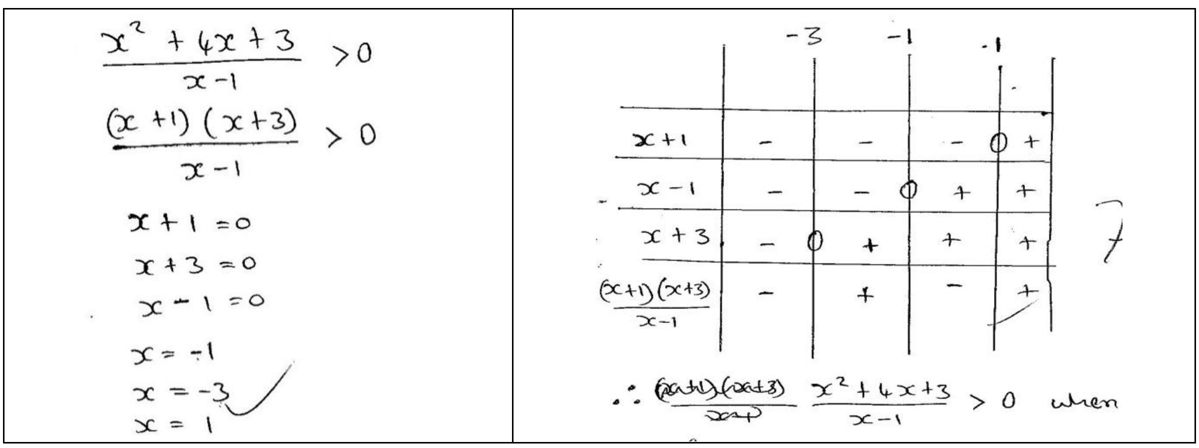


Figure 12: Yamikani’s incomplete solution process

During follow up interviews, Yamikani explained that he had not yet conceptualised how to find the range of values of x for which $\frac{x^2+4x+3}{x-1} > 0$. Others explained that “time was limited” for them to finish answering all the questions on the test. These results reflect the results of a study conducted by Godden et al. (2013).

CONCLUSION AND IMPLICATIONS

This study investigated university students’ errors when solving inequalities. Findings reveal that the identified errors were comprehension, transformational, processing skill errors and encoding errors. These findings show that absolute value inequality errors arose because of the nature of the inequalities, the lack of a formal definition of absolute values and the lack of proofs for properties of absolute inequalities. Comprehension errors indicate students’ difficulties in assigning meanings to concepts and mathematical processes related to inequalities. For instance, the results from this study show that the students replaced equality in place of inequality when solving inequality problems. Secondly, the results reveal that students in this study had difficulties with understanding the meaning of < (less than) and > (greater than) by replacing with the equality symbol in the denominator when solving a rational inequality. The generalisation of equality to inequality also suggests that “student errors are the result of previous experience in the mathematics classroom” (Radatz, 1980, p. 16). The persistence of the confusion between equation and inequality also suggests that errors are systematic; are persistent and will last for several school years, unless Mathematics educators intervene pedagogically (Radatz, 1980).

Absolute value inequality errors also reveal the students’ difficulty in assigning meaning to absolute values. These results suggest that more emphasis should be placed on absolute value inequalities of the form $|x| > M$ where $M \in \mathbb{R}^+$ and students must be made comprehend its geometrical interpretation by helping them internalise absolute value inequalities conceptually and perceive the definition of its geometrical interpretation. For instance, students should be made understand that geometrically, $|x-3| > 2$ is the set of all points whose distance from 3 is greater than 2. The compound inequality $-6 > x > 3$ shows that students would benefit more from emphasising on the concept of inequality as a propositional function, whose truth value is conditional upon the values of the variable(s). This understanding is necessary to conceive of the solution of an inequality as the set of all values of the variable(s) for which the expression becomes a true statement. This is another important aspect of algebraic thinking skills (Sierpinska et al., 2011, p. 283). The results also show that the generalisation of equality to inequality influenced some students to ignore absolute value notation when solving the absolute inequality $\frac{1}{|2x-1|} < \frac{1}{|x-2|}$ such that they solved the inequality problem as if there were no absolute values. This suggests need for Mathematics educators to design activities that would enhance students’ understanding of absolute value inequalities. Errors in the absolute value inequality of quadratic form show that the students lack the knowledge of the “why” of quadratic inequality problem solving. This result suggests need for Mathematics educators to focus on the “why” of Mathematical problem solving. Students’ incomplete solution processes suggest the need for Mathematics educators to help students understand the sign chart and represent solutions of inequalities from the sign chart.

The findings regarding representing solutions of inequalities as intervals also suggest that Mathematics educators should emphasise the fact that when writing the interval notation, students should make sure to always write the smaller value to the left and the greater value to the right. This error was also found by Bicer et al. (2014) and El-Khateeb (2016). The generalisation $|x| = x$ suggests that the educator's intention should be to make students understand the meaning of absolute values and hence absolute value inequalities. The errors made when solving the inequality $\frac{1}{|2x-1|} < \frac{1}{|x-2|}$ suggest the need for Mathematics educators to give students a clear idea of the concept of equivalent inequalities since it is this that endows the techniques of solution semantically (Blanco & Garotte, 2007). Incomplete solution processes for $\frac{x+1}{x+3} \leq -2$ and $\frac{x^2+4x+3}{x-1} < 0$ suggest that Mathematics educators need to help the students to understand that the sign chart is a horizontal number line on which we locate the zeros of the function which is on the left side of the inequality sign. Thus, the results from this study encourage educators of prerequisite mathematics courses to adopt approaches where solution techniques are conceptually connected with their theoretical underpinnings. The results also suggest that educators need to incorporate error analysis in their lesson designs, because knowledge of why students commit errors is valuable as it will help develop instructional strategies that will enhance students' conceptual understanding. Furthermore, the work of Schoenfeld and Kilpatrick (2001) seems to pull such persisting errors towards proficiency of Mathematics teaching. These findings offer a valuable opportunity for deliberations on how to prevent, limit and deal with such students' errors.

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